Step 4. Bounds on the Riemann zeta function

We wish to bound $F(s)$. We start from (2), when

$$|F(s)| \le \left|\frac{\zeta'(s)}{\zeta(s)}\right| + |\zeta(s)|\,,$$

and give upper bounds on

$$|\zeta(s)|\,, \quad |\zeta'(s)| \;\text{ and }\; \left|\frac{1}{\zeta(s)}\right|.$$

We have shown that $\zeta(s)$ has no zeros in $\operatorname{Re} s \ge 1$. We will give upper and lower bounds on $\zeta(s)$ and its derivative in the slightly larger region of

$$s = \sigma{+}it \;\text{ with }\; |t| \ge 2 \;\text{ and }\; \sigma > 1 - \frac{a}{\log|t|},$$

for any $a > 0$ as long as $\sigma > 1/2$.

Because $\zeta(\sigma - it) = \overline{\zeta(\sigma{+}it)}$ and thus

$$|\zeta(\sigma - it)| = \left|\overline{\zeta(\sigma{+}it)}\right| = |\zeta(\sigma{+}it)|\,,$$

it suffices to give bounds for t *positive*. For simplicity write $\eta(t) = a/\log t$.

4.1. Approximate $\zeta(s)$ by a finite sum.

In the next important result we approximate the Riemann zeta function by a finite sum of its Dirichlet series. First recall Theorem 6.11;

$$\sum_{1\le n\le N} \frac{1}{n^s} = 1 + \frac{1}{s{-}1} + \frac{N^{1-s}}{1{-}s} - s\int_1^N \{u\}\,\frac{du}{u^{s+1}}, \tag{28}$$

for $s \ne 1$. Let $N \to \infty$ to get (10) :

$$\zeta(s) = 1 + \frac{1}{s{-}1} - s\int_1^\infty \frac{\{u\}}{u^{1+s}}du.$$

We can only take the limit for $\operatorname{Re} s > 1$ for then $N^{1-s}/\,(1{-}s) \to 0$ as $N \to \infty$. But once the result has been proved we see that the right hand side is defined for $\operatorname{Re} s > 0$, $s \ne 1$, becoming the *definition* of the Riemann zeta function in that larger plane. If we now subtract these last two results we get

Theorem 6.24 *For all* $\operatorname{Re} s > 0$, $s \neq 1$, *and all integers* $N \geq 1$,

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + r_N(s), \tag{29}$$

where the remainder is given by

$$r_N(s) = -s \int_N^\infty \frac{\{u\}}{u^{s+1}} du \tag{30}$$

and satisfies

$$|r_N(s)| \leq |s| \int_N^\infty \frac{1}{u^{\sigma+1}} du = \frac{|s|}{\sigma N^\sigma}.$$

Note If you put $N = 1$ in Theorem 6.24 you recover Theorem 6.12 (no surprises there) while, if you let $N \to \infty$, and assume $\operatorname{Re} s > 1$ in which case

$$\lim_{N\to\infty} \frac{N^{1-s}}{s-1} = 0,$$

we recover the Dirichlet Series definition of the zeta function.

The purpose of Theorem 6.24 is to replace the infinite Dirichlet series by a finite series (called a Dirichlet Polynomial) and its strength is the ability to choose an appropriate length of polynomial N, normally *depending* on s.

4.2 Upper bound on $\zeta(s)$.

With $a > 0$ fixed and $t > 2$ we have defined $\eta(t) = a/\log t$. The important observation to make below is that for $t \geq 2$ we have

$$t^{\eta(t)} = \exp(\eta(t) \log t) = \exp\left(\frac{a}{\log t} \log t\right) = e^a,$$

a constant independent of t.

Theorem 6.25 *When* $\sigma \geq 1 - \eta(t)$, $\sigma \geq 1/2$ *and* $t \geq 2$ *we have*

$$|\zeta(\sigma+it)| \leq e^a (\log t + 5). \tag{31}$$

Proof In Theorem 6.24 with $t \geq 2$ given choose $N = [t]$, and estimate each term in (29) separately. The choice of $N = [t]$ with $t \geq 2$ implies $N \geq 2$ and $N \leq t < N + 1$. Then

$$\left|\sum_{n=1}^{N} \frac{1}{n^s}\right| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} \leq \sum_{n=1}^{N} \frac{1}{n^{1-\eta(t)}} \leq N^{\eta(t)} \sum_{n=1}^{N} \frac{1}{n} \leq e^a \sum_{n=1}^{N} \frac{1}{n},$$

since $\sigma \geq 1 - \eta(t)$ and $N \leq t$. But, a result often seen in this course, is

$$\sum_{n=1}^{N} \frac{1}{n} = 1 + \sum_{n=2}^{N} \frac{1}{n} \leq 1 + \int_1^N \frac{dt}{t} = 1 + \log N.$$

Also

$$\left|\frac{N^{1-s}}{s-1}\right| = \frac{N^{1-\sigma}}{|\sigma-1+it|} \leq \frac{N^{\eta(t)}}{|t|} \leq \frac{e^a}{2},$$

since $t \geq 2$. Finally

$$\begin{aligned} |r_N(s)| &\leq \frac{|\sigma+it|}{\sigma N^\sigma} \leq \frac{1+t/\sigma}{N^\sigma} \leq \frac{1+2t}{N^{1-\eta(t)}} \quad \text{since } \sigma \geq 1/2 \\ &\leq e^a \frac{2N+3}{N} \quad \text{since } t \leq N+1 \\ &= e^a \left(2 + \frac{3}{N}\right) \\ &\leq \frac{7}{2} e^a \end{aligned}$$

since $N \geq 2$. Combine to get the stated result. ■

Note this result, and other bounds on the Riemann zeta function require $t > 2$ (and thus $t < -2$). See the appendix for $|t| \leq 2$.

4.3 Upper bound on $\zeta'(s)$

Next we bound $|\zeta'(s)|$ from above. You can start by differentiating (28) w.r.t s. Alternatively, if you dislike differentiating under an integral you can repeat the method in Chapter 1 and apply Partial Summation in

$$\begin{aligned}\sum_{1\le n\le N}\frac{\log n}{n^s} &= \frac{N\log N}{N^s}-\int_1^N u\frac{d}{du}\left(\frac{\log u}{u^s}\right)du+\int_1^N\{u\}\frac{d}{du}\left(\frac{\log u}{u^s}\right)du\\ &= \frac{N^{1-s}\log N}{1-s}-\frac{1}{(1-s)^2}\left(N^{1-s}-1\right)+\int_1^N\{u\}\frac{d}{du}\left(\frac{\log u}{u^s}\right)du,\end{aligned}$$

after integrating by parts a number of times. Thus

$$\begin{aligned}-\sum_{n=1}^N\frac{\log n}{n^s} &= -\frac{1}{(s-1)^2}+\frac{-(1-s)N^{1-s}\log N+N^{1-s}}{(1-s)^2}\\ &\quad -\int_1^N\frac{\{u\}}{u^{1+s}}du+s\int_1^N\frac{\{u\}\log u}{u^{1+s}}du,\end{aligned}$$

for integral $N\ge 1$ and $s\ne 1$. Assume $\operatorname{Re}s>1$ and let $N\to\infty$ to get.

$$\zeta'(s)=-\frac{1}{(s-1)^2}-\int_1^\infty\frac{\{u\}}{u^{s+1}}du+s\int_1^\infty\frac{\{u\}\log u}{u^{s+1}}du,$$

which is what we would have got on differentiating (28) directly. We can see that the integrals here converge for $\operatorname{Re}s>0$.

Subtracting these last two results gives an approximation to the derivative of the Riemann zeta function by a partial sum of its Dirichlet series,

Corollary 6.26

$$\zeta'(s)=-\sum_{n=1}^N\frac{\log n}{n^s}-\frac{N^{1-s}\log N}{s-1}-\frac{N^{1-s}}{(s-1)^2}-I_1(s)+sI_2(s),\qquad(32)$$

where

$$I_1(s)=\int_N^\infty\frac{\{u\}}{u^{s+1}}du\quad\textit{and}\quad I_2(s)=\int_N^\infty\frac{\{u\}\log u}{u^{s+1}}du.$$

Leaving it to the student, each term can be estimated, giving

Theorem 6.27 *For $\sigma \geq 1 - \eta(t)$ and $t > 2$ we have*

$$|\zeta'(\sigma+it)| \leq e^a \left(\log t + 7/4\right)^2 . \tag{33}$$

Proof Exercise. ■

4.4. Upper bounds for $\operatorname{Re} s \geq 1$.

Below we use these upper bounds first for $\operatorname{Re} s > 1$. This is equivalent to choosing $a = 0$ in the results above when we then get, for $t \geq 2$,

$$|\zeta(\sigma+it)| \leq (\log t + 5) \quad \text{and} \quad |\zeta'(\sigma+it)| \leq \left(\log t + 7/4\right)^2 . \tag{34}$$

4.5. Lower bound for $\zeta(s)$.

We give an *upper bound* for $|\zeta^{-1}(\sigma+it)|$ or, equivalently, a *lower bound* for $|\zeta(\sigma+it)|$. In fact we go further and bound it **both** *away from 0* **and** to the left of the line $\operatorname{Re} s = 1$. Earlier we proved that $\zeta(s)$ is non-zero in $\operatorname{Re} s \geq 1$ but now we will have a region free of zeros to the left of $\operatorname{Re} s = 1$, i.e. a *zero-free region.*

Lemma 6.28 *For $t \geq 2$ and $2 \geq \sigma \geq 1 + \delta(t)$,*

$$|\zeta(\sigma+it)| \geq \frac{1}{2^{15}\left(\log t + 6\right)^7}, \tag{35}$$

where

$$\delta(t) = \frac{1}{2^{19}\left(\log t + 6\right)^9}. \tag{36}$$

Proof To get a lower bound in this region start from the important

$$|\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \geq 1,$$

valid for $\sigma > 1$. We can apply (34) to the $\zeta(\sigma+2it)$ term, when

$$|\zeta(\sigma+2it)| \leq \log 2t + 5 = \log t + \log 2 + 5 \leq \log t + 6,$$

say, where 6 is simply chosen as the smallest integer larger than $5 + \log 2$.

For the $\zeta(\sigma)$ term we can recall from Chapter 1 that

$$|\zeta(\sigma)| = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{\sigma}} \leq 1 + \int_1^{\infty} \frac{dy}{y^{\sigma}} = 1 + \frac{1}{\sigma-1} = \frac{\sigma}{\sigma-1} \leq \frac{2}{\sigma-1},$$

since $\sigma < 2$. Hence

$$1 \leq |\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)| \leq \left(\frac{2}{\sigma-1}\right)^3 |\zeta(\sigma+it)|^4 (\log t + 6),$$

which rearranges as

$$|\zeta(\sigma+it)| \geq \left(\frac{\sigma-1}{2}\right)^{3/4} \frac{1}{(\log t + 6)^{1/4}} \geq \left(\frac{\delta(t)}{2}\right)^{3/4} \frac{1}{(\log t + 6)^{1/4}}.$$

The result of the theorem now follows on substituting in $\delta(t)$. ■

The question you should ask, why this choice of $\delta(t)$? Answer, because of the next result. These two results can be combined as one, but since their proofs are so different I have separated them.

Theorem 6.29 *For $t \geq 2$ and $1 - \delta(t) \leq \sigma \leq 1 + \delta(t)$,*

$$|\zeta(\sigma+it)| \geq \frac{1}{2^{16} (\log t + 6)^7}.$$

Note that this is half the size of the lower bound in Lemma 6.28.

Proof Write $\sigma_t = 1 + \delta(t)$. We are assuming

$$1 - \delta(t) \leq \sigma < \sigma_t = 1 + \delta(t),$$

and so, for such σ, we have $0 < \sigma_t - \sigma \leq 2\delta(t)$.

Move along a horizontal line from $\sigma_t + it$ to $\sigma+it$. This time σ may be < 1 but since $\delta(t) \leq 1/\log t$ we can use the results of Theorem 6.27 with $a = 1$, so $|\zeta'(y + it)| \leq e (\log t + 7/4)^2$ for $y \geq 1 - 1/\log t$.

Then

$$\begin{aligned} |\zeta(\sigma+it) - \zeta(\sigma_t + it)| &= \left|\int_{\sigma_t}^{\sigma} \zeta'(y + it)\, dy\right| \leq e (\sigma_t - \sigma) (\log t + 6)^2 . \\ &\leq 2e\delta(t) (\log t + 6)^2 , \qquad (37) \end{aligned}$$

by (33), using $\log t + 7/4 \leq \log t + 6$ simply so that the bounds in (35) and (37) are *comparable*.

But how does this *upper* bound on a difference, (37) , give a *lower* bound on $\zeta(\sigma+it)$?

Idea If $w, z \in \mathbb{C}$ and $|z-w|$ is "small" then z and w are 'about' the same size. Mathematically, assume $|z-w| \leq |w|/2$. Recall the triangle inequality in the form $|a-b| \geq |a|-|b|$ for $a, b \in \mathbb{C}$ (proof $|a| = |a-b+b| \leq |a-b|+|b|$ by the 'usual' form of the triangle inequality. Rearrange to get result.) Using this

$$|z| = |w-(w-z)| \geq |w| - |w-z| \geq |w| - \frac{|w|}{2} = \frac{|w|}{2}, \tag{38}$$

i.e. we obtain a *lower* bound on $|z|$.

Apply this with $z = \zeta(\sigma+it)$ and $w = \zeta(\sigma_t + it)$. Then $|z-w| \leq |w|/2$ is satisfied if the upper bound in (37) is less than half the lower bound in (35). That is, if

$$2e\delta(t)\left(\log t + 6\right)^2 \leq \frac{1}{2}\left(\frac{\delta(t)}{2}\right)^{3/4} \frac{1}{\left(\log t + 6\right)^{1/4}}.$$

This rearranges to

$$\delta(t) \leq \frac{1}{2^{11}e^4\left(\log t + 6\right)^9},$$

which is satisfied by our choice of $\delta(t)$ in (36).

From $|z-w| \leq |w|/2$ it follows, by (38) , that $|z| \geq |w|/2$, i.e.

$$|\zeta(\sigma+it)| \geq \frac{1}{2}|\zeta(\sigma_t + it)| \geq \frac{1}{2^{16}\left(\log t + 6\right)^7} \tag{39}$$

by Lemma 6.28. ■

4.6. Upper bound on $F(s)$.

To combine the three bounds on ζ, ζ' and $1/\zeta$ they need to be comparable. For this, note that for $\sigma > 1 - 1/\log t$,

$$\begin{aligned} |\zeta(\sigma + it)| &\le e(\log t + 5) \le e(\log t + 6), \\ |\zeta'(\sigma + it)| &\le e\left(\log t + \frac{7}{4}\right)^2 \le e(\log t + 6)^2, \end{aligned}$$

are now comparable with the lower bound in Theorem 6.29. Though stated for $t > 2$ they are valid for $|t| > 2$ as long as t is replaced by $|t|$ in the bounds. Hence

Corollary 6.30 *For* $2 > \sigma \ge 1 - \delta(t)$ *and* $|t| > 2$

$$F(\sigma + it) \le 2^{19} (\log |t| + 6)^9.$$

Proof Looking back at the definition of $F(s)$,

$$\begin{aligned} |F(\sigma + it)| &\le \frac{|\zeta'(\sigma + it)|}{|\zeta(\sigma + it)|} + |\zeta(\sigma + it)| \\ &\le e(\log |t| + 6)^2 2^{16} (\log |t| + 6)^7 + (\log |t| + 6) \\ &\le 2^{19} (\log |t| + 6)^9. \end{aligned}$$

■

We in fact only want a weak version of this. For $t > 2$ we have $6 < 8.65... \times \log t$ so $\log t + 6 \le 9.65... \times \log t$ and thus

$$F(\sigma + it) \ll \log^9 |t|$$

for $t > 2$.

Theorem 6.29 implies that $\zeta(\sigma + it)$ has no zeros in the region

$$\sigma > 1 - \frac{1}{2^{19} (\log t + 6)^9}, |t| \ge 2.$$

This is called a *zero-free region*. You should draw this region to see how, the larger you take t, the less you can go to the left of the $\sigma = 1$ line. **No**

one has yet proved that there exists $\delta > 0$ such that $\zeta(s)$ has no zeros with $s : \operatorname{Re} s > 1 - \delta$.

The *Riemann Hypothesis* states that $\zeta(s)$ has no zeros with $s : \operatorname{Re} s > 1/2$. It can be shown that this is equivalent to the statement that all zeros ρ of $\zeta(s)$ which satisfy $0 < \operatorname{Re} \rho < 1$ in fact satisfy $\operatorname{Re} s = 1/2$.

Zeros with small imaginary parts.

The above results are valid for $|t| > 2$. What of $|t| \leq 2$?

On $\operatorname{Re} s = 1$ we have $\zeta(s) \neq 0$ and thus there exists $\eta > 0$ such that $|\zeta(s)| > \eta$ when $|t| < 2$. Yet $\zeta(s)$ has a continuation to the half plane $\operatorname{Re} s > 0$, $s \neq 1$ on which it is holomorphic, in particular, continuous. This means there exists $\kappa_1 > 0$ such that $|\zeta(s)| > \eta/2$ when $|t| < 2$ and $1 \geq \sigma > 1 - \kappa_1$. Similarly, it can be shown that $F(s) \ll 1$ when $|t| < 2$ and $1 \geq \sigma > 1 - \kappa_1$, provided $\kappa_1 < 1/2$. (See Additional Notes.)

It is possible, and see Jameson, Proposition 5.3.1, to prove

Proposition 6.31 *$\zeta(s)$ has no zeros in the rectangle*

$$\frac{3}{4} \leq \sigma \leq 1 \text{ and } |t| \leq \frac{5}{2}.$$

Proof Not given. ■

Divergence of $\zeta(1+it)$ for $t \neq 0$.

In Chapter 1 it was shown that the series defining $\zeta(s)$ converges absolutely for $\operatorname{Re} s > 1$. In the Problem sheet you are asked to show that the series diverges for $\operatorname{Re} s < 1$. That leaves the question of what happens **on** the vertical line $\operatorname{Re} s = 1$.

An interesting application of Theorem 6.24 is

Theorem 6.32

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+it}}$$

diverges for all $t \in \mathbb{R}$.

Proof The result if known if $t = 0$. If $t < 0$ we can look at the conjugate of the series and assume $t > 0$ as we now do.

Rearrange Theorem 6.24 as

$$\sum_{n=1}^{N} \frac{1}{n^s} = \zeta(s) - \frac{N^{s-1}}{s-1} - r_N(s),$$

where $|r_N(s)| \leq |s|/\sigma N^\sigma$. With $s = 1+it$ and $t > 0$, we have

$$\sum_{n=1}^{N} \frac{1}{n^{1+it}} = \zeta(1+it) + \frac{1}{t} e^{i(\pi/2 - t\log N)} + r_N(1+it)$$

where $|r_N(1+it)| \leq (1+|t|)/N$.

As $N \to \infty$ then $r_N(1+it) \to 0$ while the

$$\zeta(1+it) + \frac{1}{t} e^{i(\pi/2 - t\log N)}$$

are values on the circle, centre $\zeta(1+it)$, of radius $1/t$. This sequence of points do not converge but instead go forever round the circle. Hence the sequence of partial sums $\sum_{n=1}^{N} n^{-1-it}$ has no limit point as $N \to \infty$, i.e. the sequence does not converge. This is the definition of the series $\sum_{n=1}^{\infty} n^{-1-it}$ diverging. ■